Some properties of the Arnoldi based methods for linear ill-posed problems

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Abstract

In this paper we study some properties of the classical Arnoldi based methods for solving infinite dimensional linear equations involving compact operators. These problems are intrinsically ill-posed since a compact operator does not admit a bounded inverse. We study the convergence properties and the ability of these algorithms to estimate the dominant singular values of the operator.

Keywords. Linear ill-posed problem, Compact operator, Hilbert-Schmidt operator, Arnoldi algorithm, GMRES.

1 Introduction

We consider linear equations of the type

$$Af = g, \tag{1}$$

where f and g belong to a Hilbert space $\mathcal{H}$, and $A : \mathcal{H} \to \mathcal{H}$ is a compact linear operator. These kind of operators possess the general property that the spectrum $\sigma(A)$ is either finite or countably infinite; in the latter case the sequence of eigenvalues $\{\lambda_n\}_{n\geq 1}$ (arranged in order of decreasing magnitude) converges to 0. As consequence the problem (1) is ill-posed since the operator does not possess a bounded inverse. An important example of this kind of problems is provided by the Fredholm integral equation of the first kind

$$(Af)(x) = \int_{\Omega} k(x, y) f(y) dy = g(x), \tag{2}$$

where $\Omega \subseteq \mathbb{R}^q$ is open and connected. Whenever the kernel $k(x, y)$ fulfils certain hypothesis, as for instance when it has compact support, the corresponding operator A is compact on a suitable Banach space. More generally, if

$$\int_{\Omega}\int_{\Omega} |k(x, y)|^2 \, dxdy < \infty, \tag{3}$$

then $A : L^2(\Omega) \to L^2(\Omega)$ is compact.

Numerically, problems like (1) are generally faced by solving an algebraic linear system of equations

$$A_h f_h = g_h \tag{4}$$

arising from a suitable discretization of the operator A that depends on a parameter $h > 0$. Since the ill-posedness is inherited by the finite dimensional problem (4), this system is generally solved through some kind of regularization such as the popular Tikhonov method (see e.g. [10, Chapter 5] for an overview). We remark, however, that some classical iterative methods for linear systems are themselves regularizing and are sometimes used to approximate the solution of (4) without additional regularization. Among the Krylov type methods, the most highly regarded to this purpose are probably the Conjugate Gradient in the Hermitian case and the LSQR ([19]) in the general nonhermitian one, that, in exact arithmetic, is equivalent to the Conjugate Gradient applied to the normal equation (CGLS). In fact, because of the ill-conditioning of A_h, (4) is generally solved in the least-square sense, and this justifies the use of methods able to solve efficiently the normal equation. A well known property of the Conjugate Gradient method is its ability of approximating the dominating eigenvalues, that is, the dominating singular values when applied to $A_h^H A_h$. This is the main reason for which these methods are fruitfully employed to regularize an ill-posed problem.

The main drawback of the CGLS and the LSQR is that they need to work with the transpose that in some important applications is not known since A or A_h are only defined through their action. For this reason the Arnoldi based methods such as the well-known GMRES have been recently employed in this field and they have been shown to be a valid alternative to the transpose based method. In this sense, the first attempt was presented in [4]. We also quote here [6] for a recent survey. A general impression is that the Arnoldi based methods are competitive or better than the transpose based ones when the operator is nearly Hermitian or nearly normal, but definitely inferior if it is not the case. Nevertheless, some numerical experiments on highly non-normal problems like the famous equation [1] (known as BAART, see also [11])

$$\int_0^\pi \exp(x \cos y) f(y) dy = 2\frac{\sinh(x)}{x}, \quad x \in [0, \pi/2]\,, \tag{5}$$

reveal that the Arnoldi based methods are actually really competitive, both in terms of accuracy and speed.

In this paper, working in the continuous framework defined by (1), we try give a theoretical justification of some important features of the Arnoldi based methods, that are commonly considered true from experimentation. In particular, since the condition (3) implies that the operator (2) belongs to the subclass of the so called Hilbert-Schmidt operators (see [5, XI.6]), we use the properties of these operators to study the convergence rate with respect to the extendibility of the Krylov subspaces. In particular we are able to show that the rate of convergence is comparable with the rate of decay of the singular values of A. In the finite dimensional case, under special properties on the singular values, similar results were given in [18]. We remark that for linear equations of the type $(I + \lambda A)f = g$, where A is compact and $\lambda > 0$, the analysis allows to show the superlinear convergence of the residuals ([14]) that we are not able to show for problems like (1). Among the existing works in which the superlinear convergence of Krylov methods is studied in the continuous setting, we quote here the recent paper [13] and its wide bibliography. As for the finite dimensional case, we remember [22], where many Krylov methods are considered.

In this work we also show that for equations involving Hilbert-Schmidt operators the Arnoldi based methods are in fact iterative regularization approaches since the Arnoldi algorithm is able to provide, step by step, improving approximations of the dominant singular value of A. Together with the speed, this property ensures that these methods possess the basic features to be employed in the field of the regularization of certain kind of ill-posed problems.

The paper is organized as follows. In Section 2 we state the framework and the main features of the Arnoldi based methods FOM and GMRES. In Section 3 we study the convergence of the methods. The analysis is improved and extended in Section 4, where we also study the decay rate of the residual in terms of ℓ_p sequences. Section 5 is devoted to the analysis of the singular value approximations.

2 The Arnoldi based methods

Let $\mathcal{H}$ be a Hilbert space, with scalar product $<,>$ and norm $\|\cdot\|$ defined as

$$\|x\| = \langle x, x\rangle^{1/2}.$$

Throughout the paper we assume that $\mathcal{H}$ is separable, that is, it admits a countable orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$. For a given $p > 0$, we denote by ℓ_p the set of the positive sequences $\{a_j\}_{j\geq 1}$ such that

$$\sum\nolimits_{j\geq 1} a_j^p < \infty.$$

Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator. Given $g \in \mathcal{H}$, we denote by $\mathcal{K}_m = \text{span}\{g, Ag, \ldots, A^{m-1}g\}$ the Krylov subspaces generated by A and g. Setting $N = \sup_m(\dim \mathcal{K}_m)$, the Arnoldi algorithm computes an orthonormal basis $\{w_1, ..., w_m\}$ of $\mathcal{K}_m$ for each $m \leq N$. In particular, we have

$$\begin{aligned} w_1 &= \frac{g}{\|g\|}, \\ w_{m+1} &= \frac{(I - P_m)Aw_m}{\|(I - P_m)Aw_m\|}, \end{aligned}$$

where P_m is the orthogonal projection onto $\mathcal{K}_m$. If, for some m, $(I - P_m)Aw_m = 0$, then $N = m$ and $w_{N+1} = 0$. As consequence, given $g \in \mathcal{H}$ and a sequence $\{f_m\}_{m\geq 1}$, $f_m \in \mathcal{K}_m$, such that $Af_m - g \perp \mathcal{K}_m$, for $1 \leq m \leq N$, we have that $Af_N - g = 0$, so that, f_N is the solution of $Af = g$. Note that $Af_m - g \perp \mathcal{K}_m$ if and only if $P_m (Af_m - g) = 0$, that is, f_m is the solution of $P_mA|_{\mathcal{K}_m} f_m = g$. In this sense, the sequence $\{f_m\}_{m\geq 1}$ is well defined only if the operator $P_mA|_{\mathcal{K}_m} : \mathcal{K}_m \to \mathcal{K}_m$ is invertible for each $m \leq N$, and this of course depends on the properties of A. The method just described is commonly referred to as Full Orthogonalization Method ($f_0 = 0$). We recall the following basic result ([21, §1.9]).

Theorem 1 *Let $A : \mathcal{H} \to \mathcal{H}$ be a compact normal operator. Let moreover $\{\lambda_n\}_{n\in\mathbf{S}}$ be the sequence (finite $\mathbf{S} = \{1, ..., d\}$ or countably infinite $\mathbf{S} = \mathbb{N}$) of non-zero eigenvalues counted according to their multiplicities and $\{\varphi_n\}_{n\in\mathbf{S}}$ the corresponding orthonormal sequence of eigenvectors. Then*

$$Ax = \sum\nolimits_{n\in\mathbf{S}} \lambda_n \langle x, \varphi_n\rangle \varphi_n, \quad x \in \mathcal{H}. \tag{6}$$

Moreover A is self-adjoint if and only if $\lambda_n \in \mathbb{R}$, $n \in \mathbf{S}$, and is positive if and only if $\lambda_n > 0$, $n \in \mathbf{S}$.

Proposition 2 *Let $A : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint positive operator. Let $\mathcal{E}_A$ be the closed subspace of $\mathcal{H}$ generated by $\{\varphi_n\}_{n\in\mathbf{S}}$, that is, the range of A. If $g \in \mathcal{E}_A$ then the FOM approximation f_m ($f_0 = 0$) is well defined for each $m \leq N$. If dim$\mathcal{E}_A = d$ then $N \leq d$.*

Proof. Let $x \in \mathcal{H}$, $x \neq 0$. Then by (6)

$$\langle Ax, x\rangle = \sum\nolimits_{n\in\mathbf{S}} \lambda_n \left|\langle x, \varphi_n\rangle\right|^2, \tag{7}$$

so that $\langle Ax, x\rangle \geq 0$ by hypothesis. The condition $\langle Ax, x\rangle = 0$ is only possible if $\langle x, \varphi_n\rangle = 0$ for each n, that is, for $x \in \mathcal{E}_A^{\perp}$. Now taking $g \in \mathcal{E}_A$, since $\mathcal{E}_A$ is invariant with respect to A, we have that $\mathcal{K}_m(A, g) \subseteq \mathcal{E}_A$ for each m. Therefore, for $w \in \mathcal{K}_m$, $w \neq 0$, the singularity condition $P_m Aw = 0$ implies $\langle P_m Aw, w\rangle = 0$, that is, $\langle Aw, w\rangle = 0$, which contradicts what stated before. Finally, the condition $\mathcal{K}_m(A, g) \subseteq \mathcal{E}_A$ for each m, obviously yields $N \leq d$. ∎

Now assume to consider a sequence $\{f_m\}_{m\geq 1}$, $f_m \in \mathcal{K}_m$, such that the corresponding residual norm $\|Af_m - g\|$ is minimized over all the elements of $\mathcal{K}_m$, for $1 \leq m \leq N$. As before, if $N < \infty$, we have that $Af_N - g = 0$, so that, f_N is the solution of $Af = g$. Such a sequence can be constructed with the well known GMRES algorithm ($f_0 = 0$). It is also well known that $Af_m - g \perp A\mathcal{K}_m$, that is, $Q_m Af_m - g = 0$ where Q_m is the projection onto $\mathcal{K}_m$ orthogonal to $A\mathcal{K}_m$. The GMRES approximation is uniquely defined if the operator $Q_m A|_{\mathcal{K}_m} : \mathcal{K}_m \to \mathcal{K}_m$ is invertible for each $m \leq N$.

Theorem 3 *([21, Th.1.9.3]) Let $A : \mathcal{H} \to \mathcal{H}$ be a compact linear operator. Then there exists a decreasing sequence of positive real number $\{\sigma_n\}_{n\in\mathbf{S}}$ (finite or countably infinite and converging to 0) and two orthonormal sequences $\{\varphi_n\}_{n\in\mathbf{S}}$, $\{\psi_n\}_{n\in\mathbf{S}}$, such that*

$$Ax = \sum\nolimits_{n\in\mathbf{S}} \sigma_n \langle x, \varphi_n\rangle \psi_n, \quad x \in \mathcal{H}. \tag{8}$$

The sequence $\{\sigma_n\}_{n\in\mathbf{S}}$ is uniquely determined and consists of the eigenvalues of the positive self-adjoint operator $(A^*A)^{1/2}$ (the singular values of A) counted according to their multiplicities; $\{\varphi_n\}_{n\in\mathbf{S}}$ is the corresponding sequence of eigenvectors.

Remark 4 *Assuming that a compact linear operator is not of finite rank, for each $g \in \mathcal{H}$, the equation $Af = g$ has a candidate solution f given by*

$$f = \sum\nolimits_{n\geq 1} \frac{\langle g, \psi_n\rangle}{\sigma_n}\varphi_n.$$

Since $\|f\|^2 = \sum_{n\geq 1} \left|\frac{\langle g,\psi_n\rangle}{\sigma_n}\right|^2$ by Parseval identity, $f \in \mathcal{H}$ if and only if

$$\left\{\frac{|\langle g, \psi_n\rangle|}{\sigma_n}\right\}_{n\geq 1} \in \ell_2. \tag{9}$$

Assuming that $\{\sigma_n\}_{n\geq 1} \in \ell_p$, $p > 0$, by the generalized Hölder inequality (see e.g. [12, §2.7]) we have that

$$\{|\langle g, \psi_n\rangle|\}_{n\geq 1} \in \ell_{\frac{2p}{2+p}}. \tag{10}$$

Since $\frac{2p}{2+p} < p$, the condition (10) expresses what is commonly called Picard Condition, that is, the coefficients $|\langle g, \psi_n\rangle|$ must decay faster than the singular values, [10, §1.2.3].

Proposition 5 *Let $A : \mathcal{H} \to \mathcal{H}$ be a compact linear operator. Let $\mathcal{E}_{A^*A}$ be the closed subspace of $\mathcal{H}$ generated by the sequence $\{\varphi_n\}_{n\in\mathbf{S}}$ defined in (8). If $\mathcal{K}_m \subseteq \mathcal{E}_{A^*A}$ then the GMRES approximation f_m ($f_0 = 0$) is uniquely defined for each $m \leq N$. If $dim\mathcal{E}_{A^*A} = d$ then $N \leq d$.*

Proof. Assume there exists $w \in \mathcal{K}_m$, $w \neq 0$, such that $Q_m Aw = 0$. Since $Q_m Aw$ is uniquely determined by the conditions $Q_m Aw \in \mathcal{K}_m$ and $Aw - Q_m Aw \perp A\mathcal{K}_m$ we must have $Aw - Q_m Aw \perp Aw$ that leads to

$$\langle Aw, Aw\rangle = \langle Q_m Aw, Aw\rangle = 0.$$

Since

$$\langle Aw, Aw\rangle = \langle A^*Aw, w\rangle = \sum\nolimits_{n\in\mathbf{S}} \sigma_n^2 \left|\langle w, \varphi_n\rangle\right|^2,$$

this quantity is zero only if $\langle w, \varphi_n\rangle = 0$ for each $n \in \mathbf{S}$, which contradicts the hypothesis. ■

Working with normal operators the situation is simpler.

Proposition 6 *Let $A : \mathcal{H} \to \mathcal{H}$ be a compact normal operator. If $g \in \mathcal{E}_A$, the range of A, then the GMRES approximation f_m is uniquely defined for each $m \leq N$. If $dim\mathcal{E}_A = d$ then $N \leq d$.*

Proof. By (6), if $g \in \mathcal{E}_A$ then $\mathcal{K}_m \subseteq \mathcal{E}_A$ for each m. Following the proof of Proposition 5 we easily achieve the result. ■

We remark that Proposition 2 automatically states that if the operator is not of finite rank then the FOM approximation f_m ($f_0 = 0$) is well defined for each $m \leq N$. Indeed, $x \in \mathcal{E}_A^\perp$ means $x = 0$ by Parseval identity. The same consideration holds for Proposition 5.

In the remainder of the paper we always assume to work with operators whose rank is not finite, that is, $\mathbf{S} = \mathbb{N}$ in Theorems 1 and 3. All results concerning FOM and GMRES can be extended to finite rank operators under the hypotheses of Propositions 2 and 5.

3 Convergence analysis

Definition 7 *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, and let $\{\varphi_n\}_{n\in\mathbb{N}}$ be any orthonormal basis of $\mathcal{H}$. If*

$$\sum\nolimits_{n\in\mathbb{N}} \|A\varphi_n\|^2 < \infty \tag{11}$$

then A is a Hilbert-Schmidt operator.

Theorem 8 *([21, §2.4])Let $\mathcal{C}_2(\mathcal{H})$ be the set of Hilbert-Schmidt operators on $\mathcal{H}$. $\mathcal{C}_2(\mathcal{H})$ has the structure of a Hilbert space with respect to the inner product $[,]$ defined by*

$$[A, B] = trace(B^*A) := \sum\nolimits_{n\in\mathbb{N}} \langle A\varphi_n, B\varphi_n\rangle, \quad A, B \in \mathcal{C}_2(\mathcal{H}),$$

where $\{\varphi_n\}_{n\in\mathbb{N}}$ is any orthonormal basis of $\mathcal{H}$. The corresponding norm is given by

$$\|A\|_{HS}^2 = trace(A^*A) = \sum\nolimits_{n\in\mathbb{N}} \|A\varphi_n\|^2. \tag{12}$$

We observe that relation (11) ensures that a Hilbert-Schmidt operator is also compact. Indeed, by (11) and (12), each orthonormal sequence $\{\psi_n\}_{n\geq 1}$ is such that $\|A\psi_n\| \to 0$. Since a bounded linear operator A is compact if and only if $\langle A\psi_n, \psi_n\rangle \to 0$ for any orthonormal sequence $\{\psi_n\}_{n\geq 1}$ ([21, Th.1.8.7]), the statement follows from the Cauchy-Schwartz inequality. Observe moreover that since

$$trace(A^*A) = \sum_{j\in\mathbb{N}} \sigma_j^2,$$

by (12) we have that for each orthonormal sequence $\{\psi_n\}_{n\geq 1}$ it holds

$$\sum_{n\geq 1} \|A\psi_n\|^2 \leq \sum_{j\in\mathbb{N}} \sigma_j^2. \tag{13}$$

Theorem 9 *Let $A \in \mathcal{C}_2(\mathcal{H})$ with a singular value expansion (8). If g satisfies the condition (9) then $\|f_m - f\| \to 0$. Moreover, there exists a non negative sequence $\{a_i\}_{i\geq 1} \in \ell_2$, such that*

$$\|Af_m - g\| \leq \left(\sum_{i>m} a_i^2\right)^{1/2}. \tag{14}$$

Proof. Since the GMRES minimizes the residual in $\mathcal{K}_m$ we have

$$\|Af_m - g\| \leq \|AP_m f - g\|.$$

Moreover $P_m f \to f$ as $m \to \infty$ and thus we have $\|Af_m - g\| \to 0$. Since the solution is unique, $\|f_m - f\| \to 0$. For the second part, let $\{z_i\}_{i\in\mathbb{N}}$ be a basis of $\mathcal{H}$ such that $\{z_1, ..., z_m\}$ is an orthonormal system for $A\mathcal{K}_m$. We have $\langle Af_m - g, z_i\rangle = 0$ for $i = 1, ..., m$. Hence

$$\begin{aligned} Af_m - g &= \sum_{i>m} \langle Af_m - g, z_i\rangle z_i \\ &= \sum_{i>m} \langle f_m - f, A^* z_i\rangle z_i, \end{aligned}$$

and so

$$\begin{aligned} \|Af_m - g\|^2 &= \sum_{i>m} |\langle f_m - f, A^* z_i\rangle|^2 \\ &\leq \|f_m - f\|^2 \sum_{i>m} \|A^* z_i\|^2. \end{aligned}$$

Since $\|f_m - f\| \to 0$ for $m \to \infty$, by (13) and since A^* is still Hilbert-Schmidt we obtain the result taking $a_i := \|f_m - f\| \|A^* z_i\|$. ■

4 Extendibility of the Krylov subspaces

The connection between the residuals of FOM and GMRES, expressed by the famous peak-plateau phenomenon (see e.g. [3]), ensures that the GMRES convergence implies the FOM convergence. This means that the sequence of FOM approximations is bounded, that is, we have $\|f_m\| \leq M$. As consequence, we have that the FOM residual (and hence the GMRES one) is bounded by $Mh_{m+1,m}$, where $h_{m+1,m} := \langle w_{m+1}, Aw_m \rangle = \|(I - P_m)Aw_m\|$. Indeed, since the FOM approximation satisfies $P_m (Af_m - g) = 0$, and $P_m g = g$, we have

$$\begin{aligned} Af_m - g &= (I - P_m)Af_m \\ &= (I - P_m)A \sum_{i=1}^{m} \langle f_m, w_i \rangle w_i \\ &= (I - P_m)A \langle f_m, w_m \rangle w_m, \end{aligned}$$

because $P_m Aw_i = Aw_i$ for $i = 1, ..., m - 1$. Therefore

$$\|Af_m - g\| = h_{m+1,m} |\langle f_m, w_m \rangle| \leq h_{m+1,m} \|f_m\| . \tag{15}$$

We start with the following result.

Lemma 10 *([17, p. 125]). Let* $A : \mathcal{H} \to \mathcal{H}$ *be a bounded linear operator with singular values* $\{\sigma_j\}_{j\in\mathbb{N}}$*. Let* $\{g_1, ..., g_n\}$ *and* $\{h_1, ..., h_n\}$ *be any pair of finite orthonormal systems in* $\mathcal{H}$*. Then*

$$|\det [\langle g_i, Ah_j \rangle]| \leq \prod_{j=1}^{n} \sigma_j .$$

As shown in [14], since the matrix $[\langle w_{i+1}, Aw_j \rangle]$ is upper triangular we have that

$$\prod_{j=1}^{n} h_{j+1,j} = \det [\langle w_{i+1}, Aw_j \rangle] \leq \prod_{j=1}^{n} \sigma_j . \tag{16}$$

Definition 11 *Let* $A : \mathcal{H} \to \mathcal{H}$ *be a compact operator and let* $p > 0$*. Then* A *is* p*-nuclear and we write* $A \in \mathcal{C}_p(\mathcal{H})$ *if* $\{\sigma_j\}_{j\in\mathbb{N}} \in \ell_p$*.*

The above definition implies that Hilbert-Schmidt operators are 2-nuclear operators. In this situation the self adjoint positive operator A^*A is 1-nuclear, since $\{\sigma_j\}_{j\in\mathbb{N}} \in \ell_2$ implies $\{\sigma_j^2\}_{j\in\mathbb{N}} \in \ell_1$. We remark that the class $\mathcal{C}_1(\mathcal{H})$ is often called trace-class whereas $\mathcal{C}_p(\mathcal{H})$, $p \geq 1$, is also called von Neumann-Schatten class (see [21, Ch.2]).

Assuming that $A \in \mathcal{C}_p(\mathcal{H})$, $p > 0$, by the arithmetic-geometric mean inequality

$$\left(\prod_{j=1}^{n} h_{j+1,j}\right)^{p/n} \leq \frac{\sum_{j\geq 1} \sigma_j^p}{n} . \tag{17}$$

Under the hypothesis of $0 < p \leq 1$, a very similar result ([18, Th. 3.1]) can be derived using [15, Th. 5.8.10]. Here we can state the following.

Proposition 12 *Let* $A \in \mathcal{C}_p(\mathcal{H})$ *with* $p \geq 1$*. Then* $\{h_{j+1,j}\}_{j\in\mathbb{N}} \in \ell_p$*.*

Proof. Therefore, let $\{\varphi_n\}_{n\in\mathbb{N}}$ be the orthonormal basis of eigenvectors of A^*A. Let moreover H be the positive square root of A^*A, defined by

$$Hx = \sum_{n\in\mathbb{N}} \sigma_n \langle x, \varphi_n \rangle \varphi_n, \quad x \in \mathcal{H},$$

so that we can consider the polar decomposition $A = UH$, where U is a partial isometry [5, p.935]. For any orthonormal sequence $\{z_j\}_{j\geq 1}$ we have

$$\left\| H^{1/2} z_j \right\|^2 = \langle z_j, H z_j \rangle = \sum\nolimits_{i\geq 1} \sigma_i \left| \langle z_j, \varphi_i \rangle \right|^2,$$

where $H^{1/2}$ is the positive square root of H. Setting q be such that

$$1 = \frac{1}{p} + \frac{1}{q},$$

using Hölder inequality we have

$$\begin{aligned} \left\| H^{1/2} z_j \right\|^2 &= \sum\nolimits_{i\geq 1} \sigma_i \left| \langle z_j, \varphi_i \rangle \right|^{2/p} \left| \langle z_j, \varphi_i \rangle \right|^{2/q} \\ &\leq \left(\sum\nolimits_{i\geq 1} \sigma_i^p \left| \langle z_j, \varphi_i \rangle \right|^2 \right)^{1/p} \left(\sum\nolimits_{i\geq 1} \left| \langle z_j, \varphi_i \rangle \right|^2 \right)^{1/q} \\ &\leq \left(\sum\nolimits_{i\geq 1} \sigma_i^p \left| \langle z_j, \varphi_i \rangle \right|^2 \right)^{1/p}. \end{aligned}$$

Thus

$$\begin{aligned} \sum\nolimits_{j\geq 1} \left\| H^{1/2} z_j \right\|^{2p} &\leq \sum\nolimits_{j\geq 1} \sum\nolimits_{i\geq 1} \sigma_i^p \left| \langle z_j, \varphi_i \rangle \right|^2 \\ &= \sum\nolimits_{i\geq 1} \sigma_i^p \sum\nolimits_{j\geq 1} \left| \langle z_j, \varphi_i \rangle \right|^2 \\ &\leq \sum\nolimits_{i\geq 1} \sigma_i^p. \end{aligned} \tag{18}$$

Now,

$$\begin{aligned} \sum\nolimits_{j\geq 1} h_{j+1,j}^p &= \sum\nolimits_{j\geq 1} \langle w_{j+1}, U H w_j \rangle^p \\ &= \sum\nolimits_{j\geq 1} \left\langle H^{1/2} U^* w_{j+1}, H^{1/2} w_j \right\rangle^p \\ &\leq \sum\nolimits_{j\geq 1} \left\| H^{1/2} U^* w_{j+1} \right\|^p \left\| H^{1/2} w_j \right\|^p \\ &\leq \left(\sum\nolimits_{j\geq 1} \left\| H^{1/2} U^* w_{j+1} \right\|^{2p} \right)^{1/2} \left(\sum\nolimits_{j\geq 1} \left\| H^{1/2} w_j \right\|^{2p} \right)^{1/2}. \end{aligned}$$

Since both $\{w_j\}_{j\geq 1}$ and $\{U^* w_{j+1}\}_{j\geq 1}$ are orthonormal systems the result follows from (18). ■

The above proposition show the connection between the extendibility of the Krylov subspaces and the singular values, for $p\,(\geq 1)$-nuclear operators. The case $0 < p < 1$ is more difficult to study since the Hölder inequality is reversed (see e.g [12, §2.8]) and we cannot arrive at (18). We need some additional hypothesis as stated by Proposition 14.

Lemma 13 *[2, p.259]Let $\{a_j\}_{j\geq 1}, \{b_j\}_{j\geq 1}$ be non increasing sequences of real numbers such that $\sum_{j=1}^n a_j \leq \sum_{j=1}^n b_j$ for each $n \geq 1$. Then, for any convex function Φ, that is,*

$$\Phi(\alpha t + (1-\alpha)u) \leq \alpha \Phi(t) + (1-\alpha)\Phi(u), \quad t, u \in \mathbb{R}, \alpha \in (0,1),$$

and such that $\Phi(t) \to 0$ as $t \to -\infty$, we have $\sum_{j=1}^n \Phi(a_j) \leq \sum_{j=1}^n \Phi(b_j)$

Proposition 14 *Let $A \in \mathcal{C}_p(\mathcal{H})$ with $p > 0$. If $\{h_{j+1,j}\}_{j\in\mathbb{N}}$ is non increasing then $\{h_{j+1,j}\}_{j\in\mathbb{N}} \in \ell_p$. Moreover if*

$$h_{j+2,j+1} \leq \left(\prod_{i=1}^{j} h_{i+1,i}\right)^{1/j}, \quad j \in \mathbb{N}, \tag{19}$$

then $\{h_{j+1,j}\}_{j\in\mathbb{N}} \in \ell_{p+\varepsilon}$ for each $\varepsilon > 0$.

Proof. By (16) we have that for each $n \geq 1$

$$\sum_{j=1}^{n} \log h_{j+1,j} \leq \sum_{j=1}^{n} \log \sigma_j. \tag{20}$$

Therefore, if $\{h_{j+1,j}\}_{j\in\mathbb{N}}$ is non increasing, applying Lemma 13 with $\Phi(t) = \exp(pt)$ we obtain the result. The weaker hypothesis (19) ensures that the sequence

$$s_j := \left(\prod_{i=1}^{j} h_{i+1,i}\right)^{1/j}$$

is non increasing. Moreover by (16) and using the arithmetic-geometric mean inequality we obtain

$$\begin{aligned}\prod_{j=1}^{n} s_j &\leq \prod_{j=1}^{n} \left(\prod_{i=1}^{j} \sigma_i\right)^{1/j} \\ &\leq \prod_{j=1}^{n} \left(\frac{1}{j}\sum_{i=1}^{j} \sigma_i^p\right)^{1/p} \\ &\leq \prod_{j=1}^{n} \left(\frac{C}{j}\right)^{1/p}\end{aligned}$$

where $C = \sum_{j\geq 1} \sigma_j^p$. Working as before with $\Phi(t) = \exp(\overline{p}t)$, $\overline{p} > p$, we have that $\{s_j\}_{j\in\mathbb{N}} \in \ell_{\overline{p}}$ and hence the result. ∎

In Figure 1 we compare the behavior of the sequences $\{h_{j+1,j}\}_{j\in\mathbb{N}}$ and $\{\sigma_j\}_{j\in\mathbb{N}}$ for the problem (5) in which the singular values decay exponentially (that is, $\{\sigma_j\}_{j\in\mathbb{N}} \in \ell_p$ for each $p > 0$). In the same figure we also consider the compact self-adjoint operator defined by the kernel

$$k(x,y) = \begin{cases} x(y-1), & x < y, \\ y(x-1), & x \geq y, \end{cases} \quad x, y \in [0,1], \tag{21}$$

which represents the Green function for the second derivative (see [11]). The singular values are $\sigma_j = (j\pi)^{-2}$ and hence the corresponding sequence is ℓ_p for each $p > 1/2$.

Theorem 15 *Let $A \in \mathcal{C}_p(\mathcal{H})$, $p > 0$. If the condition (9) is satisfied then for the GMRES residual it holds*

$$\{\|Af_m - g\|\}_{m\geq 1} \in \ell_p. \tag{22}$$

Proof. Since the convergence is ensured by the Picard Condition (9), by (15) we have that

$$\|Af_m - g\| \leq C h_{m+1,m}.$$

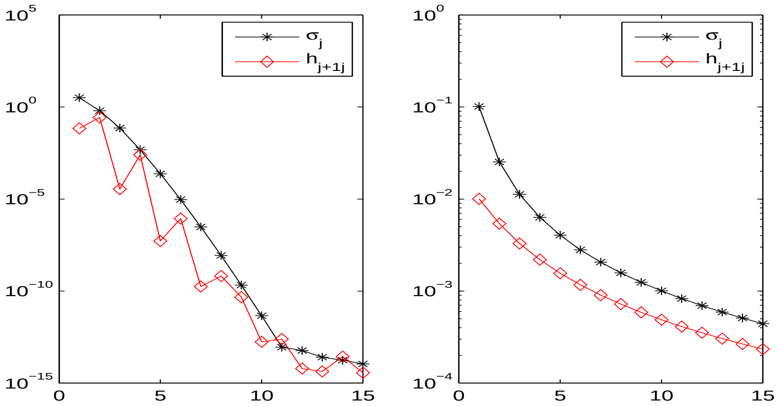

Figure 1: Decay behavior of the sequences $\{h_{j+1j}\}_{j\in\mathbb{N}}$ and $\{\sigma_j\}_{j\in\mathbb{N}}$ for the problem (5) on the left, and (21) on the right.

Therefore, by (16)

$$\prod_{m=1}^{n} \|Af_m - g\| \leq C^n \prod_{m=1}^{n} \sigma_m.$$

Since $\{\|Af_m - g\|\}_{m\geq 1}$ is non increasing, the result follows immediately from Lemma 13. ■

The theoretical analysis given in this section does not allow to state the q-superlinear convergence of the Arnoldi based methods. Indeed, by (22) we have only proved that the rate of convergence is equal to the decay rate of the singular values. On the other side it is well known in literature that for problems like $(I + \lambda A)f = g$ $(\lambda > 0)$ the q-superlinear convergence is ensured (see e.g. [14]). We can explain this basic difference in the following way. Denoting by $H_m^s \in \mathbb{C}^{m\times m}$ the Hessenberg matrix whose entries $h_{i,j}$ are given $h_{i,j} = \langle w_i, Aw_j\rangle$, it is known that the FOM residual can also be written as

$$\|Af_m - g\| = h_{m+1,m} \left| e_m^H (H_m^s)^{-1} e_1 \right|,$$

where e_i is the i-th element of the canonical base in $\mathbb{C}^m$. It is known from [16] that H_m^s is nonderogatory, that is, the minimal polynomial $q(z)$ is the characteristic polynomial, so that we can write

$$(H_m^s)^{-1} = -\left(\sum_{j=0}^{m-1} \alpha_{j+1} (H_m^s)^j\right) \frac{1}{\alpha_0}, \tag{23}$$

which arises from the equation

$$0 = q(H_m^s) = \alpha_0 I + \alpha_1 H_m^s + \ldots + \alpha_m (H_m^s)^m.$$

Since $\alpha_m = 1$, $\alpha_0 = (-1)^m \det(H_m^s)$, and exploiting the Hessenberg structure of H_m^s, which yields

$$\begin{aligned} e_m^H (H_m^s)^k e_1 &= 0, \quad k = 0, \ldots, m-2, \\ e_m^H (H_m^s)^{m-1} e_1 &= \prod_{j=1}^{m-1} h_{j+1,j}, \end{aligned}$$

we finally obtain

$$\|Af_m - g\| = \frac{\prod_{j=1}^m h_{j+1,j}}{|\det(H_m^s)|}. \tag{24}$$

Since $\operatorname{span}\{g, Ag, \ldots, A^{j-1}g\} = \operatorname{span}\{g, (I+\lambda A)g, \ldots, (I+\lambda A)^{j-1}g\}$, the numerator of (24) does not depend on the invertibility of the operator that defines the equation (A or $I + \lambda A$), but only on the extendibility of the Krylov subspaces. On the other side, the matrix H_m^s is expected to retain the spectral properties of A or $I + \lambda A$, and hence, only in the latter case we can state the existence of a constant C such that $1/|\det(H_m^s)| \le C$ (at least for m large enough). Such a bound immediately yields the q-superlinear convergence for nuclear operators by using (17).

5 The approximation of the singular values

We denote by $H_m \in \mathbb{C}^{(m+1)\times m}$ the matrix containing H_m^s with entries $h_{i,j} = \langle w_i, Aw_j\rangle$, $i = 1, ..., m+1$, $j = 1, ..., m$. We have

$$Aw_k = \sum\nolimits_{i=1}^{k+1} h_{i,k} w_i, \quad k \le m. \tag{25}$$

Let us consider the SVD factorization of H_m, that is, for $1 \le k \le m$ we consider the equations

$$H_m \varphi_k^{(m)} = \sigma_k^{(m)} \psi_k^{(m)}, \tag{26}$$

$$H_m^* \psi_k^{(m)} = \sigma_k^{(m)} \varphi_k^{(m)}, \tag{27}$$

where $\left\{\sigma_k^{(m)}\right\}_{k=1,\ldots,m}$ are the singular values, arranged in decreasing order, and where $\psi_k^{(m)} \in \mathbb{C}^{m+1}$, $\varphi_k^{(m)} \in \mathbb{C}^m$, are the k-th left and right singular vector respectively. Denoting by $\psi_{kj}^{(m)}$ and $\varphi_{kj}^{(m)}$ the j-th entry of the vectors $\psi_k^{(m)}$ and $\varphi_k^{(m)}$, we can state the following result, proved in [18] in the finite dimensional case.

Proposition 16 *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator. Let moreover*

$$u_k^{(m)} = \sum\nolimits_{j=1}^{m+1} w_j \psi_{kj}^{(m)}, \quad k \le m+1,$$

$$v_k^{(m)} = \sum\nolimits_{j=1}^{m} w_j \varphi_{kj}^{(m)}, \quad k \le m.$$

Then

$$Av_k^{(m)} - \sigma_k^{(m)} u_k^{(m)} = 0, \tag{28}$$

$$P_m(A^* u_k^{(m)} - \sigma_k^{(m)} v_k^{(m)}) = 0, \tag{29}$$

$$P_m A^* u_{m+1}^{(m)} = 0. \tag{30}$$

Proof. (28) follows directly by (25), (26), and using the Hessenberg structure of H_m. Indeed we have

$$
\begin{aligned}
Av_k^{(m)} &= \sum\nolimits_{j=1}^{m} Aw_j \varphi_{kj}^{(m)} \\
&= \sum\nolimits_{j=1}^{m} \sum\nolimits_{i=1}^{j+1} h_{i,j} w_i \varphi_{kj}^{(m)} \\
&= \sum\nolimits_{i=1}^{m+1} w_i \sum\nolimits_{j=1}^{m} h_{i,j} \varphi_{kj}^{(m)} \\
&= \sum\nolimits_{i=1}^{m+1} w_i \sigma_k^{(m)} \psi_{ki}^{(m)} \\
&= \sigma_k^{(m)} u_k^{(m)}.
\end{aligned}
$$

Moreover, since $h_{i,j} = \langle w_i, Aw_j \rangle = \langle A^* w_i, w_j \rangle$, for $j \le m$ the j-th entry of $H_m^* \psi_k^{(m)}$ is given by

$$
\begin{aligned}
\sum\nolimits_{i=1}^{m+1} \overline{h_{j,i}} \psi_{ki}^{(m)} &= \sum\nolimits_{i=1}^{m+1} \overline{\langle w_j, A^* w_i \rangle} \psi_{ki}^{(m)} \\
&= \left\langle A^* u_k^{(m)}, w_j \right\rangle.
\end{aligned}
$$

Then, using (27) we have that

$$
\begin{aligned}
\left\langle A^* u_k^{(m)}, w_j \right\rangle &= \sigma_k^{(m)} \varphi_{kj}^{(m)} \\
&= \sigma_k^{(m)} \left\langle v_k^{(m)}, w_j \right\rangle,
\end{aligned}
$$

which yields (29) by

$$
\left\langle A^* u_k^{(m)} - \sigma_k^{(m)} v_k, w_j \right\rangle = 0, \quad j \le m.
$$

Finally, (30) follows directly by $H_m^* \psi_{m+1}^{(m)} = 0$. ■

Observe that by Parseval identity $\left\| u_k^{(m)} \right\| = \left\| \psi_k^{(m)} \right\|_2 = 1$, $k \le m+1$, and $\left\| v_k^{(m)} \right\| = \left\| \varphi_k^{(m)} \right\|_2 = 1$, $k \le m$. Moreover by Parseval equation

$$
\begin{aligned}
\left\langle u_i^{(m)}, u_k^{(m)} \right\rangle &= \sum\nolimits_{j=1}^{m+1} \left\langle u_i^{(m)}, w_j \right\rangle \left\langle w_j, u_k^{(m)} \right\rangle \\
&= \sum\nolimits_{j=1}^{m+1} \psi_{ij}^{(m)} \overline{\psi_{kj}^{(m)}} \\
&= \delta_{ik}.
\end{aligned}
$$

Analogously $\left\langle v_i^{(m)}, v_k^{(m)} \right\rangle = \delta_{ik}$. In this view, equations (28)-(29) state that the triplets $\left(\sigma_k^{(m)}, u_k^{(m)}, v_k^{(m)}\right)$, $1 \le k \le m$, that can be generated by the Arnoldi algorithm, are worth of further investigation in order to understand if we are able to construct an approximation of the expansion (8).

Theorem 17 *Let $A \in \mathcal{C}_2(\mathcal{H})$. Then for each fixed k*

$$
\left\| A^* u_k^{(m)} - \sigma_k^{(m)} v_k^{(m)} \right\| \to 0, \quad m \to \infty.
$$

Proof. By the definition of $v_k^{(m)}$ in Proposition 16 we have that $v_k^{(m)} \in \mathcal{K}_m$ and hence, by (29), $\sigma_k^{(m)} v_k^{(m)}$ is the orthogonal projection of $A^* u_k^{(m)}$ onto $\mathcal{K}_m$, i.e.,

$$P_m(A^* u_k^{(m)}) = \sigma_k^{(m)} v_k^{(m)}.$$

Therefore

$$\begin{aligned} \left\| A^* u_k^{(m)} - \sigma_k^{(m)} v_k^{(m)} \right\|^2 &= \left\| A^* u_k^{(m)} \right\|^2 - \left\| \sigma_k^{(m)} v_k^{(m)} \right\|^2 \\ &= \sum_{i \geq m+1} \left| \left\langle A^* u_k^{(m)}, w_i \right\rangle \right|^2 \\ &= \sum_{i \geq m+1} \left| \left\langle u_k^{(m)}, A w_i \right\rangle \right|^2 \\ &\leq \sum_{i \geq m+1} \left\| A w_i \right\|^2 . \end{aligned} \tag{31}$$

Thus, by (13), we have that $\left\| A^* u_k^{(m)} - \sigma_k^{(m)} v_k \right\|^2$ is bounded by the tail of a convergent series and hence the result follows. ■

In the above theorem we have assumed that the orthonormal system $\{w_1, ..., w_m\}$ that represent an orthonormal basis of $\mathcal{K}_m$, generates the whole space $\mathcal{H}$ as $m \to \infty$. Proposition 16 and Theorem 17 states that the sequence $\left\{ \sigma_k^{(m)} \right\}_{m \geq 1}$ tends to a certain singular value of A as $m \to \infty$ but we cannot be sure that this singular value is exactly σ_k. In order to fix the problem, we need to show that each σ_k converges to a singular value of H_m as $m \to \infty$. This is proved by the following, whose consequence is the one to one correspondence between the dominant singular values of A and the ones of H_m.

Theorem 18 *Let $A \in \mathcal{C}_2(\mathcal{H})$ with a singular value expansion given by (8). For a fixed k let moreover*

$$\varphi_{kj}^{(m)} = \langle w_j, \varphi_k \rangle , \quad j \leq m.$$

Then

$$\left\| H_m^* H_m v_k^{(m)} - \sigma_k^2 v_k^{(m)} \right\|_2 \to 0, \quad m \to \infty.$$

Proof. By (8) we have

$$A^* A \varphi_k = \sigma_k^2 \varphi_k. \tag{32}$$

Writing

$$\varphi_k = \sum_{j \geq 1} \langle \varphi_k, w_j \rangle w_j ,$$

then substituting in (32) and splitting the sum, we obtain

$$\sum_{j=1}^m \langle \varphi_k, w_j \rangle A^* A w_j + \sum_{j \geq m+1} \langle \varphi_k, w_j \rangle A^* A w_j = \sigma_k^2 \sum_{j=1}^m \langle \varphi_k, w_j \rangle w_j + \sigma_k^2 \sum_{j \geq m+1} \langle \varphi_k, w_j \rangle w_j .$$

Therefore, for $i \leq m$,

$$\sum_{j=1}^m \langle \varphi_k, w_j \rangle \langle w_i, A^* A w_j \rangle + \sum_{j \geq m+1} \langle \varphi_k, w_j \rangle \langle w_i, A^* A w_j \rangle = \sigma_k^2 \langle w_i, \varphi_k \rangle .$$

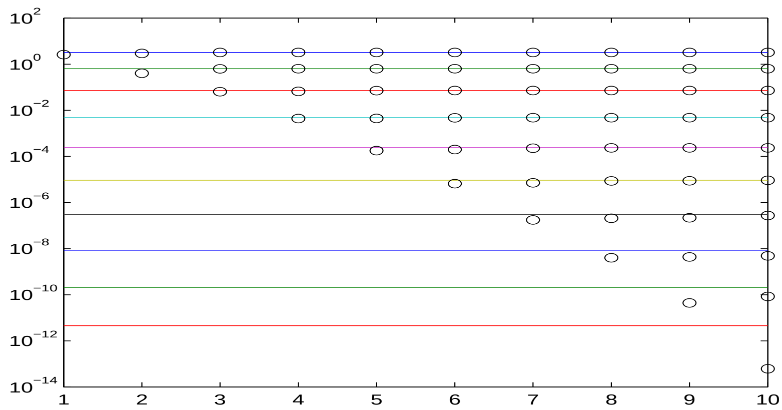

Figure 2: Plot of the singular values of the matrix H_m versus the iteration number m, for the problem (5). The solid lines represent the singular values of the operator.

For the second term on the left we have

$$\begin{aligned}\left|\sum\nolimits_{j\geq m+1}\langle\varphi_k,w_j\rangle\langle w_i,A^*Aw_j\rangle\right| &\leq \sum\nolimits_{j\geq m+1}|\langle\varphi_k,w_j\rangle|\,|\langle w_i,A^*Aw_j\rangle| \\ &\leq \left(\sum\nolimits_{j\geq m+1}|\langle\varphi_k,w_j\rangle|^2\right)^{1/2}\left(\sum\nolimits_{j\geq m+1}|\langle w_i,A^*Aw_j\rangle|^2\right)^{1/2} \\ &\leq \left(\sum\nolimits_{j\geq m+1}\|A^*Aw_j\|^2\right)^{1/2},\end{aligned}$$

that goes to 0 as $m\to\infty$ since A^*A is still Hilbert-Schmidt. Therefore we have proved that

$$\sum\nolimits_{j=1}^{m}\langle\varphi_k,w_j\rangle\langle w_i,A^*Aw_j\rangle\to\sigma_k^2\langle w_i,\varphi_k\rangle,$$

as $m\to\infty$. Since $\langle w_i,A^*Aw_j\rangle$ is just the (i,j) entry of the matrix $H_m^*H_m$ the result follows. ■

In Figure 2 we show the convergence of the singular values for the problem (5).

Proposition 19 *Let $A\in\mathcal{C}_2(\mathcal{H})$ with a singular value expansion given by (8). Then for each fixed k, $\left|\left\langle u_k^{(m)},\psi_k\right\rangle\right|\to 1$ and $\left|\left\langle v_k^{(m)},\varphi_k\right\rangle\right|\to 1$ as $m\to\infty$.*

Proof. By (8) we have

$$Av_k^{(m)}=\sum\nolimits_{n\geq 1}\sigma_n\left\langle v_k^{(m)},\varphi_n\right\rangle\psi_n.$$

Moreover by (28)

$$\begin{aligned}Av_k^{(m)} &= \sigma_k^{(m)}u_k^{(m)} \\ &= \sigma_k^{(m)}\sum\nolimits_{n\geq 1}\left\langle u_k^{(m)},\psi_n\right\rangle\psi_n,\end{aligned}$$

so that, for each n,

$$\sigma_n \left\langle v_k^{(m)}, \varphi_n \right\rangle = \sigma_k^{(m)} \left\langle u_k^{(m)}, \psi_n \right\rangle . \tag{33}$$

At the same time

$$\begin{aligned} A^* u_k^{(m)} &= \sum\nolimits_{n\geq 1} \left\langle u_k^{(m)}, \psi_n \right\rangle A^* \psi_n \\ &= \sum\nolimits_{n\geq 1} \sigma_n \left\langle u_k^{(m)}, \psi_n \right\rangle \varphi_n, \end{aligned}$$

and

$$\sigma_k^{(m)} v_k^{(m)} = \sigma_k^{(m)} \sum\nolimits_{n\geq 1} \left\langle v_k^{(m)}, \varphi_n \right\rangle \varphi_n.$$

By (29) we then have

$$\begin{aligned} A^* u_k^{(m)} - \sigma_k^{(m)} v_k^{(m)} &= \sum\nolimits_{n\geq 1} \left(\left\langle u_k^{(m)}, \psi_n \right\rangle \sigma_n - \left\langle v_k^{(m)}, \varphi_n \right\rangle \sigma_k^{(m)} \right) \varphi_n \\ &= \sum\nolimits_{i\geq m+1} q_i w_i, \end{aligned}$$

where

$$q_i = \left\langle A^* u_k^{(m)} - \sigma_k^{(m)} v_k^{(m)}, w_i \right\rangle .$$

Since $q_i \to 0$ as $m \to \infty$ by Theorem 17, uniformly with respect to n, we have that for each n

$$\left\langle u_k^{(m)}, \psi_n \right\rangle \sigma_n = \left\langle v_k^{(m)}, \varphi_n \right\rangle \sigma_k^{(m)} + c_m, \quad c_m \to 0. \tag{34}$$

Now, joining (33) and (34) we find

$$\left\langle u_k^{(m)}, \psi_n \right\rangle \sigma_n^2 = \left\langle u_k^{(m)}, \psi_n \right\rangle \left(\sigma_k^{(m)} \right)^2 + c_m \sigma_n.$$

Since $\sigma_k^{(m)} \to \sigma_k$, and since the above equation is true for each n we have $\left\langle u_k^{(m)}, \psi_n \right\rangle \to 0$ for $k \neq n$ and $m \to \infty$ and consequently

$$\left| \left\langle u_k^{(m)}, \psi_k \right\rangle \right| \to 1.$$

Analogously one shows that $\left| \left\langle v_k^{(m)}, \varphi_k \right\rangle \right| \to 1$. ■

Proposition 19 shows that the vectors $u_k^{(m)}$ ($v_k^{(m)}$) tends to $e^{i\theta}\psi_k$ ($e^{i\theta}\varphi_k$) for a certain angle θ. In Figure 3, using again the model problem (5) we can observe the convergence to 1 of the sequences $\left\{ \left| \left\langle v_k^{(m)}, \varphi_k \right\rangle \right| \right\}_{m\geq 1}$ and $\left\{ \left| \left\langle u_k^{(m)}, \psi_k \right\rangle \right| \right\}_{m\geq 1}$ for $k = 1, 2, 3$.

Joining the results of this section, we have that if A is a Hilbert-Schmidt operator, then for each fixed k and up to a given phase angle, $\left(\sigma_k^{(m)}, v_k^{(m)}, u_k^{(m)} \right) \to (\sigma_k, \varphi_k, \psi_k)$ as $m \to \infty$. This means that in general we cannot expect that for a given m, $\left(\sigma_k^{(m)}, v_k^{(m)}, u_k \right) \approx (\sigma_k, \varphi_k, \psi_k)$ for $k = 1, ..., m$ but only for $k \ll m$, where the distance between k and m depends on the rate of decay of the singular values. In any case we can expect that for $k \ll m$ the Arnoldi algorithm is able to approximate the truncated expansion

$$\sum\nolimits_{j=1}^{k} \sigma_j \left\langle \cdot, \varphi_j \right\rangle \psi_j \approx \sum\nolimits_{j=1}^{k} \sigma_j^{(m)} \left\langle \cdot, v_j^{(m)} \right\rangle u_j^{(m)},$$

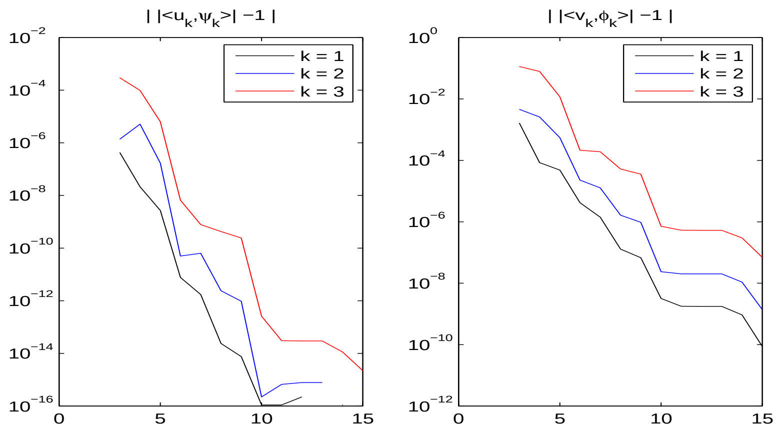

Figure 3: Convergence of the singular first three singular vectors with respect to the dimension of the Krylov subspaces for problem (5).

and consequently that

$$\left\| A - \sum_{j=1}^{k} \sigma_j^{(m)} \left\langle \cdot, v_j^{(m)} \right\rangle u_j^{(m)} \right\| \approx \sigma_{k+1}.$$

Remark 20 *An interesting consequence of the results of this section is that for Hilbert-Schmidt operators* $\|H_m\| \to \|A\|$ *even if* A *is highly non normal.*

5.1 The self-adjoint case

Under the hypothesis that the operator A is self-adjoint, in order to state the convergence of the singular values of H_m we do not require that A is Hilbert-Schmidt.

Theorem 21 *Let* $A : \mathcal{H} \to \mathcal{H}$ *be a self-adjoint compact operator. Under the hypothesis of Proposition 16, for each fixed* k

$$\left\| Au_k^{(m)} - \sigma_k^{(m)} v_k^{(m)} \right\| \to 0, \quad m \to \infty. \tag{35}$$

Proof. Following the proof of Theorem 17 we have

$$\begin{aligned}
\left\|Au_k^{(m)} - \sigma_k^{(m)} v_k^{(m)}\right\|^2 &= \sum\nolimits_{i\geq m+1} \left|\left\langle Au_k^{(m)}, w_i\right\rangle\right|^2 \\
&= \sum\nolimits_{i\geq m+1} \left|\sum\nolimits_{j=1}^{m+1} \psi_{kj}^{(m)} \left\langle Aw_j, w_i\right\rangle\right|^2 \\
&= \sum\nolimits_{i\geq m+1} \left|\sum\nolimits_{j=1}^{m+1} \psi_{kj}^{(m)} \overline{h_{i,j}}\right|^2 \\
&\leq \sum\nolimits_{i\geq m+1} \left|\left(\sum\nolimits_{j=1}^{m+1} \left|\psi_{kj}^{(m)}\right|^2\right)^{1/2} \left(\sum\nolimits_{j=1}^{m+1} \left|h_{i,j}\right|^2\right)^{1/2}\right|^2 \\
&= \sum\nolimits_{i\geq m+1} \left|\left(\sum\nolimits_{j=1}^{m+1} \left|h_{i,j}\right|^2\right)\right| \\
&= \left|h_{m+1,m}\right|^2 + \left|h_{m+1,m+1}\right|^2 + \left|h_{m+2,m+1}\right|^2, \qquad (36)
\end{aligned}$$

where the last equality follows from the Hessenberg structure of H_m. At this point since $h_{m+1,m} \to 0$ as $m \to \infty$ when working with compact operator (the proof follows the one of [21, Th.1.8.7] with a slight modification) and since, under the same hypothesis

$$h_{m+1,m+1} = \langle w_{m+1}, Aw_{m+1}\rangle \to 0,$$

we obtain the result ■

Observe that since $h_{m+1,m} = \overline{h_{m,m+1}}$, by (36) and $|\langle w_i, Aw_j\rangle| \leq \|Aw_j\|$ we have

$$\left\|Au_k^{(m)} - \sigma_k^{(m)} v_k\right\|^2 \leq 3\left\|Aw_{m+1}\right\|^2. \qquad (37)$$

By comparing this bound with (31) we can understand that for self-adjoint Hilbert-Schmidt operators the convergence of the singular values can be very fast. In order to avoid repetitions we omit the proof of the following result, that is essentially based on the convergence stated by (35).

Proposition 22 *The results of Theorem 18 and Proposition 19 remain true when $A : \mathcal{H} \to \mathcal{H}$ is a self-adjoint compact operator.*

In Figure 4 we show the convergence of the singular values for the self-adjoint operator defined by (21).

5.2 A further note on the GMRES residual

Using the singular value analysis just presented, we can state the following (cf [8]).

Proposition 23 *For the GMRES residual it holds*

$$\|Af_m - g\| = \left|\left\langle g, u_{m+1}^{(m)}\right\rangle\right|, \qquad (38)$$

where $u_{m+1}^{(m)}$ is defined as in Proposition 16.

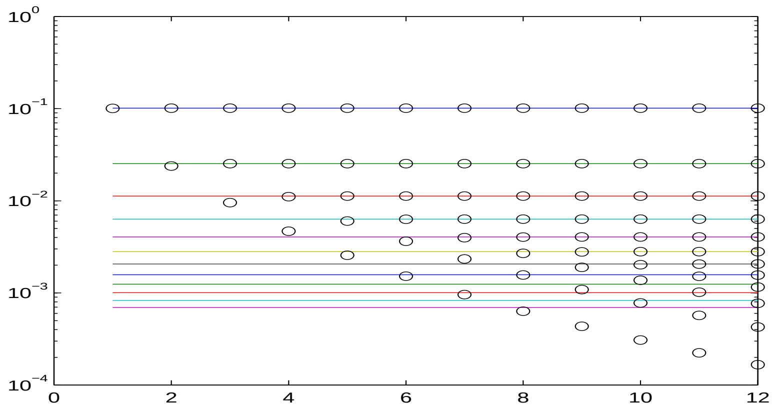

Figure 4: Plot of the singular values of the matrix H_m versus the iteration number m, for the problem (21). The solid lines represent the singular values of the operator.

Proof. Since $Af_m - g \in \mathcal{K}_{m+1} = \text{span}\{u_1^{(m)}, \ldots, u_{m+1}^{(m)}\}$ we can write

$$Af_m - g = \sum_{i=1}^{m+1} \left\langle Af_m - g, u_i^{(m)} \right\rangle u_i^{(m)}. \tag{39}$$

Moreover, using the condition $Af_m - g \perp A\mathcal{K}_m$, and taking $\left\{v_i^{(m)}\right\}_{i=1,\ldots,m}$ as an orthonormal basis for $\mathcal{K}_m$, we have

$$\left\langle Af_m - g, Av_i^{(m)} \right\rangle = \sigma_i^{(m)} \left\langle Af_m - g, u_i^{(m)} \right\rangle = 0, \quad i = 1, ..., m.$$

Using the above relation in (39) we obtain

$$Af_m - g = \left\langle Af_m - g, u_{m+1}^{(m)} \right\rangle u_{m+1}^{(m)}.$$

The result then follows from the expansion

$$\begin{aligned} Af_m &= A \sum_{i=i}^{m} \left\langle f_m, v_i^{(m)} \right\rangle v_i^{(m)} \\ &= \sum_{i=i}^{m} \sigma_i^{(m)} \left\langle f_j, v_i^{(m)} \right\rangle u_i^{(m)}, \end{aligned}$$

which yields

$$\left\langle Af_m, u_{m+1}^{(m)} \right\rangle = 0.$$

■

Observe that by Proposition 19 we can expect that

$$\|Af_m - g\| \approx |\langle g, \psi_{m+1} \rangle| \, ,$$

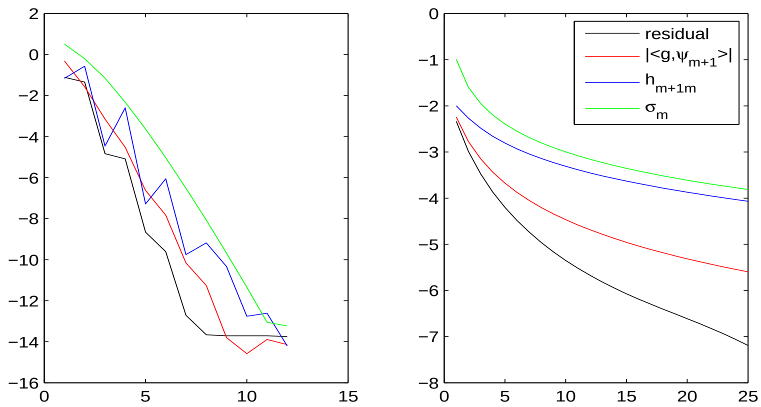

Figure 5: GMRES residual history for the problem (5) on the left, and (21) on the right.

that is, $\{\|Af_m - g\|\}_{m\geq 1}$ is close to a $\ell_{\frac{2p}{2+p}}$ sequence (cf. (10)). Similar arguments, but with a completely different approach, where used in [8] in the finite dimensional case. Formula (38) is also interesting since it allows to compare the GMRES with the truncated singular value decomposition, TSVD. Indeed, the m-th TSVD approximation is given by

$$f_m^{TSVD} = \sum_{j=1}^{m} \frac{\langle g, \psi_j \rangle}{\sigma_j} \varphi_j,$$

so that

$$\left\| Af_m^{TSVD} - g \right\| = \left(\sum_{j\geq m+1} |\langle g, \psi_j \rangle|^2 \right)^{1/2}.$$

In Figure 5 we report two experiments in which we plot the GMRES residual, comparing it with the sequences $\{\sigma_m\}_{m\geq 1}$, $\{h_{m+1,m}\}_{m\geq 1}$ and $\{|\langle g, \psi_{m+1} \rangle|\}_{m\geq 1}$. The comparisons (even on other problems not reported) confirm the theoretical analysis of Section 4, and reveal that the sequence $\{|\langle g, \psi_{m+1} \rangle|\}_{m\geq 1}$ well represents the GMRES behavior, especially for problems where the singular values decay exponentially.

6 Conclusion

The results exposed in this paper represent a theoretical justification of some important properties of the Arnoldi based methods already observed experimentally. We refer in particular to [6, 7, 18], where many experiments concerning the rate of convergence of the Arnoldi methods and the SVD approximation have been presented on some classical linear ill-posed problems. While not considered in this paper, the use of the Arnoldi algorithm for solving the Tikhonov minimization is fully justified for linear equations involving Hilbert-Schmidt operators, especially for what concerns the parameter choice rule such as the L-curve analysis [9], the Generalized Cross Validation, or the

Reginska criterium [20]. Indeed, the efficiency of these techniques is closely related to the efficient approximation of the dominating singular values of the underlying operator.

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